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Refinable functions with non-integer dilations

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Abstract

Refinable functions and distributions with integer dilations have been studied extensively since the pioneer work of Daubechies on wavelets. However, very little is known about refinable functions and distributions with non-integer dilations, particularly concerning its regularity. In this paper we study the decay of the Fourier transform of refinable functions and distributions. We prove that uniform decay can be achieved for any dilation. This leads to the existence of refinable functions that can be made arbitrarily smooth for any given dilation factor. We exploit the connection between algebraic properties of dilation factors and the regularity of refinable functions and distributions. Our work can be viewed as a continuation of the work of Erdös [P. Erdös, On the smoothness properties of a family of Bernoulli convolutions, Amer. J. Math. 62 (1940) 180–186], Kahane [J.-P. Kahane, Sur la distribution de certaines séries aléatoires, in: Colloque de Théorie des Nombres, Univ. Bordeaux, Bordeaux, 1969, Mém. Soc. Math. France 25 (1971) 119–122 (in French)] and Solomyak [B. Solomyak, On the random series $\sum \pm\lambda^n$ (an Erdös problem), Ann. of Math. (2) 142 (1995) 611–625] on Bernoulli convolutions. We also construct explicitly a class of refinable functions whose dilation factors are certain algebraic numbers, and whose Fourier transforms have uniform decay. This extends a classical result of Garsia [A.M. Garsia, Arithmetic properties of Bernoulli convolutions, Trans. Amer. Math. Soc. 102 (1962) 409–432].

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1. Introduction

In this paper we study the refinement equation

$$f(x)=\sum_{j=0}^{m}c_j f(\lambda x-d_j),\qquad \sum_{j=0}^{m}c_j=|\lambda|, \tag{1.1}$$

where $\lambda\in\mathbb{R}$ with $|\lambda|>1$ and all c_j, d_j are real. It is well known that up to a scalar multiple the above refinement equation has a unique distribution solution f, which is furthermore compactly supported. We shall refer to the distribution solution $f(x)$ of (1.1) with $\widehat{f}(0)=1$ *the solution* to (1.1). For the refinement equation (1.1), the value λ is called the *dilation factor* of the refinement equation, and $\{d_j\}$ the *translation set* or simply the *translations*. The coefficients $\{c_j\}$ are called the *weights* (even though they can be negative). For simplicity we shall call a solution $f(x)$ to (1.1) a *λ-refinable distribution (function) with translations $\{d_j\}$*.

The questions we study concern the regularity of λ-refinable functions or distributions. Particularly we are interested in refinable functions whose dilation factors are non-integers. For example, is it possible to find a $3/2$-refinable function that is smooth? More generally, is it possible to find a smooth λ-refinable function for any λ with $|\lambda|>1$?

Refinable functions play a fundamental role in many areas such as the construction of compactly supported wavelets, self-affine tiles, and in the study of subdivisions schemes in approximation theory, see e.g. Daubechies [4], Lagarias and Wang [12] and Cavaretta et al. [1]. In all cases the dilation factors are restricted to integers or integral matrices, as are the translations. It is well known that for any integer dilation λ there exist λ-refinable functions with integer translations that can be made arbitrarily smooth. The simplest example is the B-spline $B_m(x)$, which is obtained by convolving $\chi_{[0,1)}$ with itself m times. B_m is λ-refinable for any integer λ, $|\lambda|>1$. The B-splines have important applications in subdivision schemes and computer aided geometric designs. With integer dilations and translations one may impose strong constraints on the weights while still attaining smoothness. The most important example is the construction of a class of smooth refinable functions whose integer translates are mutually orthogonal that began with the seminal work of Daubechies [3] leading to compactly supported orthonormal wavelets.

But the regularity question becomes more complicated, and perhaps more interesting from the pure analysis point of view, when the dilation factors λ are non-integers, particularly when the translations are still restricted to integers. There is a strong connection with number theory that still needs to be fully exploited. The regularity of refinable functions and distributions seem to be strongly affected by algebraic properties of the dilation factors.

One way to characterize regularity is to consider the decay of $\widehat{f}$. Let $f(x)$ be a distribution. We say $\widehat{f}$ has *uniform decay at infinity* if $\widehat{f}(\xi)=O(|\xi|^{-\gamma})$ for some $\gamma>0$. Suppose that $\widehat{f}$ has uniform γ-decay at infinity. Let $f^{*n}(x):=f*f*\cdots*f(x)$ in which f convolves with itself $n-1$ times. Then $\widehat{f^{*n}}=\widehat{f}^n$, which has uniform $n\gamma$-decay at infinity. By taking n large one can make f^{*n} an arbitrarily smooth function. Furthermore, if f is λ-refinable then so is f^{*n}. In fact if f is λ-refinable with integer translations then so is f^{*n}. Thus we shall focus on the following question. Given any $\lambda\in\mathbb{R}$ with $|\lambda|>1$, is there a λ-refinable distribution $f(x)$ such that $\widehat{f}$ has uniform decay at infinity? What if the translations are required to be integers?

A large amount of literature has been devoted to this questions in the case of *Bernoulli convolutions*, which are the solutions to

$$f(x) = \frac{|\lambda|}{2} f(\lambda x) + \frac{|\lambda|}{2} f(\lambda x - 1). \tag{1.2}$$

Many of these studies apply to the more general setting of refinable functions with integer translations as well. Erdös [5] proved that under the integer translations setting any λ-refinable distribution f has $\widehat{f}(\xi) \nrightarrow 0$ as $|\xi| \to \infty$ if λ is a Pisot number, i.e. an algebraic integer whose algebraic conjugates are all inside the unit circle. This immediately implies that f cannot be in L^1. It remains an open question whether Pisot numbers are the only dilations for which one cannot construct L^1 refinable functions with integer translations. Also under the integer translation setting Kahane [11] proved that $\widehat{f}$ does not have uniform decay at infinity for any λ-refinable distribution if λ is a Salem number, i.e. an algebraic integer whose algebraic conjugates are all inside or on the unit circle, assuming that some conjugates lie actually on the unit circle. (Both Erdös and Kahane established their results for Bernoulli convolutions, but with some technical twisting we may extend their results to the more general setting, see Appendix A.) In the positive direction, Garsia [9] proved that the Bernoulli convolution $f(x)$ of (1.2) is in L^∞, if the dilation λ is an algebraic integer whose algebraic conjugates are all outside the unit circle and the constant term for its minimal polynomial is ± 2. Garsia's result remains today as the only explicitly known class of Bernoulli convolutions that are in L^1. Feng and Wang [8] explicitly constructed a large class of algebraic integer dilations λ for which the corresponding Bernoulli convolutions are not in L^2. In the generic setting Solomyak [16] proved that for almost all dilations $\lambda \in (1, 2)$ the corresponding Bernoulli convolution is in L^1, and more recently, Peres and Schlag [13] proved that the Fourier transform of the Bernoulli convolution has uniform decay at infinity for almost all dilations $\lambda \in (1, 2)$. It is not clear whether the latter result holds for almost all $\lambda \in (1, \infty)$. Our results in this paper can be viewed as an extension of the aforementioned studies.

When the dilations and translations are both integers and the weights are nonnegative, the uniform decay property can be characterized completely.

Theorem 1.1. *Let $f(x)$ be the distribution solution to the refinement equation*

$$f(x) = \sum_{j=0}^{m} c_j f(\lambda x - d_j), \qquad \sum_{j=0}^{m} c_j = |\lambda|,$$

where $\lambda \in \mathbb{Z}$ and $c_j > 0$, $d_j \in \mathbb{Z}$ for all j. Then the following are equivalent:

(A) *$\widehat{f}$ has uniform decay at ∞.*
(B) $f \in L^\infty(\mathbb{R})$.
(C) $f \in L^1(\mathbb{R})$.
(D) $\lim_{|\xi| \to \infty} \widehat{f}(\xi) = 0$.
(E) *$\widehat{f}(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.*
(F) *For any $n \in \mathbb{Z} \setminus \{0\}$ there exists a $k = k(n) > 0$ such that $P(\lambda^{-k} n) = 0$, where $P(\xi) := \frac{1}{|\lambda|} \sum_{j=0}^{m} c_j e(-d_j \xi)$ with $e(t) := e^{2\pi i t}$.*

The trigonometric polynomial $P(\xi)$ is called the *symbol* of the refinement equation. Note that Theorem 1.1 can be partially extended to the case of rational dilations.

Theorem 1.2. *Let $f(x)$ be the distribution solution to the refinement equation*

$$f(x)=\sum_{j=0}^{m} c_j f(\lambda x-d_j), \quad \sum_{j=0}^{m} c_j=|\lambda|,$$

where $\lambda\in\mathbb{Q}$, $|\lambda|>1$ and $c_j>0$, $d_j\in\mathbb{Z}$ for all j. Suppose that for any $n\in\mathbb{Z}\setminus\{0\}$ there exists a $k=k(n)>0$ such that $P(\lambda^{-k}n)=0$, where $P(\xi)$ is the symbol of the refinement equation. Then the following hold:

(A) *$\widehat{f}$ has uniform decay at ∞.*
(B) *$f\in L^\infty(\mathbb{R})$.*
(C) *$\widehat{f}(n)=0$ for all $n\in\mathbb{Z}\setminus\{0\}$.*

We remark that both Theorems 1.1 and 1.2 hold under the weaker assumption that $|P(\xi)|<1$ for all $\xi\in\mathbb{R}\setminus\mathbb{Z}$. This is rather easy to see from the proof. Also, Theorem 1.2 holds for any dilation λ, but unless $\lambda^k\in\mathbb{Q}$ for some $k>0$ there exists no refinement equations satisfying the hypotheses of the theorem.

One of the well-known questions concerning refinable functions is whether one can construct arbitrarily smooth refinable functions for rational dilations such as $\lambda=3/2$. Here we answer this question.

Corollary 1.3. *Let $\lambda=p/q$ where $p>|q|$ are integers and $(p,q)=1$. Then the refinable distribution satisfying*

$$f(x)=\frac{1}{|q|}\sum_{j=0}^{p-1} f\left(\frac{p}{q}x-j\right)$$

is in $L^\infty(\mathbb{R})$, and $\widehat{f}$ has uniform decay at ∞. As a consequence, for any $k\geqslant 0$ there exists a compactly supported λ-refinable function f with integer translations and nonnegative weights such that f is in C^k.

In Section 2 we shall explicitly construct a differentiable 3/2-refinable function.

Theorem 1.4. *Let $\lambda\in\mathbb{R}$ with $|\lambda|>1$ and $c_j>0$ for all j. Let $f=f_t$ be the distribution solution of the refinement equation*

$$f(x)=\sum_{j=0}^{m} c_j f(\lambda x-d_j), \quad \sum_{j=0}^{m} c_j=|\lambda|, \tag{1.3}$$

where d_0, d_1 are fixed and distinct, and $d_m=t$ with t being a parameter. Then there exists an $E:=E_\lambda\subset\mathbb{R}$ independent of $\{c_j\}_{j=0}^m$ with $\dim_{\mathrm{H}}(E)=0$, such that $\widehat{f_t}$ has uniform decay at infinity for each $t\in\mathbb{R}\setminus E$.

Here $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension (see e.g. [7] for a definition). Obviously the set E has Lebesgue measure 0. The other translations $d_3,\dots,d_{m-1}$ may or may not depend on t. By taking convolution of $f_t(x)$ with itself repeatedly we easily obtain the following corollary.

Corollary 1.5. *Let $\lambda \in \mathbb{R}$ with $|\lambda| > 1$. Then for any $k \geqslant 0$ there exists a compactly supported λ-refinable function with nonnegative weights that is in C^k.*

Note that in the language of self-similar measures the above corollary states that for any λ with $|\lambda| > 1$ there exists a self-similar measure with contraction λ^{-1} whose density can be arbitrarily smooth. Our next theorem is an extension of Garsia [9].

Theorem 1.6. *Let $p(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be irreducible (but not necessarily monic) such that all roots of $p(x)$ are outside the unit circle. Let λ be a real root of $p(x)$ and $f(x)$ be the distribution solution to*

$$f(x) = \frac{|\lambda|}{|a_0|} \sum_{j=0}^{|a_0|-1} f(\lambda x - j). \tag{1.4}$$

Then $\widehat{f}$ has uniform decay at infinity. Furthermore, $f \in L^\infty$.

Garsia [9] proved that f is L^∞ in the case $a_n = 1$ and $a_0 = \pm 2$. No uniform decay property was established in [9], however.

2. Proof of Theorems 1.1 and 1.2

We consider in this section refinable distributions with integer translations and integer or rational dilations. First we introduce a notation. For any $x \in \mathbb{R}$ we let $\|x\|_{\mathbb{Z}}$ denote the distance of x to the integers $\mathbb{Z}$. Thus $\|x\|_{\mathbb{Z}} \leqslant 1/2$. We may without loss of generality consider the refinement equation

$$f(x) = \sum_{j=0}^{m} c_j f(\lambda x - d_j), \qquad \sum_{j=0}^{m} c_j = |\lambda|, \tag{2.1}$$

where $\lambda \in \mathbb{Z}$ or $\lambda \in \mathbb{Q}$, $c_j > 0$ for all j and $0 = d_0 < d_1 < \cdots < d_m$ are in $\mathbb{Z}$. Furthermore, we assume that $\gcd(d_1, d_2, \ldots, d_m) = 1$.

Lemma 2.1. *Let $\lambda = p/q \in \mathbb{Q}$ with $p > |q|$ and $(p, q) = 1$. Let $Q(t)$ be a trigonometric polynomial with $Q(0) = 1$ and $|Q(t)| < 1$ for any $t \notin \mathbb{Z}$. Furthermore, $Q(t)$ has the property that for any $n \in \mathbb{Z} \setminus \{0\}$ there exists a $k > 0$ such that $Q(\lambda^{-k} n) = 0$. Fix an $\varepsilon > 0$. Suppose that $|t| > \varepsilon$ and $\|t\|_{\mathbb{Z}} < \varepsilon$. Then there exists an $\ell \in \mathbb{N}$ such that $|g(t)| \leqslant C\varepsilon|\lambda|^{-\ell}|g(\lambda^{-\ell} t)|$, where $g(t) := \prod_{j=1}^{\infty} Q(\lambda^{-j} t)$ and $C = \max|Q'(t)|$. In particular, g has uniform decay at infinity.*

Proof. Write $t = n + \delta$ where $n \in \mathbb{Z}$ and $|\delta| = \|t\|_{\mathbb{Z}}$. Note that $|t| > \varepsilon$, so $n \neq 0$. Therefore there exists an $\ell > 0$ such that $Q(\lambda^{-\ell} n) = 0$. This means that

$$\left|Q\left(\lambda^{-\ell} t\right)\right| = \left|Q\left(\lambda^{-\ell} t\right) - Q\left(\lambda^{-\ell} n\right)\right| \leqslant C\left|\lambda^{-\ell}\delta\right| < C\varepsilon|\lambda|^{-\ell}.$$

Therefore

$$\left|g(t)\right| = \prod_{j=1}^{\infty}\left|Q\left(\lambda^{-j} t\right)\right| \leqslant Q\left(\lambda^{-\ell} t\right) \prod_{j=\ell}^{\infty}\left|Q\left(\lambda^{-j} t\right)\right| \leqslant C\varepsilon|\lambda|^{-\ell}\left|g\left(\lambda^{-\ell} t\right)\right|.$$

The decay property of g can now be established. Let $M = \sup\{|Q(t)|\colon \|t\|_{\mathbb{Z}} \geqslant \varepsilon\} < 1$. For any $t \in \mathbb{R}$ with $|t| > |\lambda|$, we can uniquely write $t = \lambda^N s$ for some s with $|s| \in [1, |\lambda|]$ and $N \in \mathbb{N}$. Now g is an analytic function, and so it is bounded on $[-|\lambda|, |\lambda|]$, say by the constant $K > 0$. Suppose that $\|\lambda^{-1}t\|_{\mathbb{Z}} \geqslant \varepsilon$. Then $|Q(\lambda^{-1}t)| \leqslant M$. Hence $|g(t)| = |g(\lambda^N s)| \leqslant M|g(\lambda^{N-1}s)|$. On the other hand, suppose that $\|\lambda^{-1}t\|_{\mathbb{Z}} < \varepsilon$. Then

$$\bigl|g(t)\bigr| \leqslant \bigl|g\bigl(\lambda^{-1}t\bigr)\bigr| \leqslant C\varepsilon|\lambda|^{-\ell}\bigl|g\bigl(\lambda^{N-\ell-1}s\bigr)\bigr|. \tag{2.2}$$

Take $\rho \in (M, 1)$ such that $\rho^{k+1} \geqslant C\varepsilon|\lambda|^{-k}$ for all $k \in \mathbb{N}$. This ρ clearly exists if we take ε to be small enough. Then (2.2) becomes $|g(t)| \leqslant \rho^{\ell+1}|g(\lambda^{N-\ell-1}s)|$. Combining with the case $\|\lambda^{-1}t\|_{\mathbb{Z}} \geqslant \varepsilon$ we now have $|g(t)| = |g(\lambda^N s)| \leqslant \rho^N K$. The uniform decay property is now established by taking $\gamma = \log\rho^{-1}/\log|\lambda|$, and $|g(\lambda^N s)| \leqslant K|\lambda|^{-\gamma N}$. □

Proof of Theorem 1.1. It is clear that (B) ⇒ (C) ⇒ (D).

(D) ⇒ (E). We have $\widehat{f}(\xi) = \widehat{f}(0)\prod_{j=1}^{\infty} P(\lambda^{-j}\xi)$. In particular $\widehat{f}(\lambda^k\xi) = \widehat{f}(\xi)\prod_{j=0}^{k-1} P(\lambda^j\xi)$. This implies that $\widehat{f}(\lambda^k n) = \widehat{f}(n)$. By letting $k \to \infty$ we have $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

(E) ⇒ (F). Again we invoke the property $\widehat{f}(\xi) = \widehat{f}(\lambda^{-k}\xi)\prod_{j=1}^{k} P(\lambda^{-j}\xi)$. Since $\widehat{f}(0) \neq 0$ and $\widehat{f}(\xi)$ is analytic, it follows from $\widehat{f}(n) = 0$ that $\prod_{j=1}^{k} P(\lambda^{-j}n) = 0$ when k is large enough. (F) now follows.

(F) ⇒ (B). By our convention it is assumed that $\widehat{f}(0) = 1$. It is known that f is in fact a probability measure, see e.g. Falconer [7]. Now f is compactly supported. Define $F(x) = \sum_{n\in\mathbb{Z}} f(x-n)$. Then F is a Radon measure, and hence a distribution. We have $F = f * \delta_{\mathbb{Z}}$ where $\delta_{\mathbb{Z}} := \sum_{n\in\mathbb{Z}} \delta(x-n)$. The Poisson Summation Formula yields $\widehat{F} = \widehat{f} \cdot \widehat{\delta_{\mathbb{Z}}} = \widehat{f} \cdot \delta_{\mathbb{Z}}$. Therefore $\widehat{F} = \delta$, and hence $F = 1$. This implies that $f \in L^{\infty}$.

(A) ⇒ (D). This is clear.

(F) ⇒ (A). $P(\xi)$ satisfies the hypotheses of Lemma 2.1. By the lemma $\widehat{f}(\xi) = \prod_{j=1}^{\infty} P(\lambda^{-j}\xi)$ has uniform decay at infinity. □

Proof of Theorem 1.2. The proof is identical to the proof of Theorem 1.1, and we only give a brief explanation. Clearly, the hypotheses of the theorem implies (C), that is, $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. The argument used to prove (F) ⇒ (B) in Theorem 1.1 now applies to prove that $f \in L^{\infty}$. The uniform decay of $\widehat{f}$ is established by Lemma 2.1. □

Proof of Corollary 1.3. Let $P(\xi) = \frac{1}{p}\sum_{j=0}^{p-1} e(-j\xi)$ be the symbol of the refinement equation. $P(\xi)$ clearly satisfies the hypotheses of Theorem 1.2. Hence $\widehat{f}$ has uniform decay at infinity, and $f \in L^{\infty}$. Convolving f with itself sufficiently many times yields a λ-refinable function that can be made arbitrarily smooth. □

Example 2.1. We consider the compactly supported refinable function $f(x)$ given by

$$f(x) = \frac{1}{2}f\left(\frac{3}{2}x\right) + \frac{1}{2}f\left(\frac{3}{2}x + 1\right) + \frac{1}{2}f\left(\frac{3}{2}x - 1\right).$$

f is in L^{∞} by Corollary 1.3, and $\widehat{f}$ has uniform decay. We prove that $\widehat{f}(\xi) = O(|\xi|^{-1})$ as $|\xi| \to \infty$. This immediately implies that f^{*n} is $(n-2)$-times differentiable.

To prove this, observe that the symbol of the refinement equation is $P(\xi)=\frac{1+2\cos(2\pi\xi)}{3}$. Let $\lambda=3/2$ and $\xi=\lambda^N s$ where $|s|\in[1,\lambda)$. We prove that $|\widehat{f}(\xi)|\leqslant K\lambda^{-N}$ for $K=\max_{|s|\leqslant\lambda}|\widehat{f}(s)|$. A simple check with Maple shows that by taking $\varepsilon=\frac{1.05}{2\pi}$ we have $M=\max_{\|t\|_{\mathbb{Z}}\geqslant\varepsilon}|P(t)|<\lambda^{-1}=2/3$. Furthermore, another check with Maple shows that $|P(\frac{n}{3}+t)|\leqslant C|t|$ for some C with $C\varepsilon<\lambda^{-1}=2/3$, as long as $n\neq 3k$ and $|t|<\varepsilon$. As a result, by the estimations used to prove Lemma 2.1, we obtain $|f(\lambda^N s)|\leqslant K\lambda^{-N}=K(\frac{2}{3})^N$.

3. Proof of Theorem 1.4

Without loss of generality we may assume that (1.3) is normalized so that $d_0=0$ and $d_1=1$. Let $P_t(\xi)=\frac{1}{|\lambda|}\sum_{j=0}^m c_j e(-d_j\xi)$. We have

$$P_t(\xi)=\frac{1}{|\lambda|}\left(c_0+c_1e(-\xi)+c_me(-t\xi)+\sum_{j=2}^{m-1}c_je(-d_j\xi)\right),$$

and $\widehat{f_t}(\xi)=\prod_{j=1}^{\infty}P_t(\lambda^{-j}\xi)$. To prove Theorem 1.4 we establish a series of lemmas.

Lemma 3.1. *Let $\lambda>1$. For a fixed t assume that $\widehat{f_t}$ has no uniform decay at infinity. Then for any $\varepsilon,\delta>0$ and $N\in\mathbb{N}$, there exist $a\in[1,\lambda)$ and $n>N$ such that both $\{0\leqslant j\leqslant n-1\colon\ \|a\lambda^j\|_{\mathbb{Z}}\geqslant\delta\}$ and $\{0\leqslant j\leqslant n-1\colon\ \|at\lambda^j\|_{\mathbb{Z}}\geqslant\delta\}$ have cardinality less than εn.*

Proof. Assume the lemma is false. Then there exist ε,δ and N such that $\{0\leqslant j\leqslant n-1\colon\ \|a\lambda^j\|_{\mathbb{Z}}\geqslant\delta\}$ or $\{0\leqslant j\leqslant n-1\colon\ \|at\lambda^j\|_{\mathbb{Z}}\geqslant\delta\}$ have cardinality at least εn for all $a\in[1,\lambda]$ and $n>N$. Let

$$M=M_\delta=\max\bigl\{\bigl|P_t(\xi)\bigr|\colon\ \|\xi\|_{\mathbb{Z}}\geqslant\delta\text{ or }\|t\xi\|_{\mathbb{Z}}\geqslant\delta\bigr\}.$$

It is clear $0<M<1$. Let

$$\mathcal{A}_n:=\bigl\{0\leqslant j\leqslant n-1\colon\ \bigl\|a\lambda^j\bigr\|_{\mathbb{Z}}\geqslant\delta\text{ or }\bigl\|at\lambda^j\bigr\|_{\mathbb{Z}}\geqslant\delta\bigr\}.$$

Set $h=\frac{\varepsilon\log(1/M)}{\log\lambda}$.

Now any $\xi>0$ can be uniquely written as $\xi=\lambda^n s$ for some $s\in[1,\lambda)$ and $n\in\mathbb{Z}$. For $n>N$ we have

$$\bigl|\widehat{f_t}(\xi)\bigr|=\prod_{j=0}^{n-1}\bigl|P_t\bigl(\lambda^j s\bigr)\bigr|\cdot\bigl|\widehat{f_t}(s)\bigr|\leqslant C\prod_{j\in\mathcal{A}_n}\bigl|P_t\bigl(\lambda^j s\bigr)\bigr|\leqslant CM^{\varepsilon n}=C\lambda^{-hn},$$

where $C=\max_{s\in[1,\lambda]}|\widehat{f_t}(s)|$. This shows decay in $\widehat{f_t}(\xi)$ for $\xi>0$. However, $\widehat{f_t}(-\xi)=\overline{\widehat{f_t}(\xi)}$, which also yields decay in $\widehat{f_t}(\xi)$ for $\xi<0$. Hence $\widehat{f_t}(\xi)$ has uniform decay at ∞. This is a contradiction. $\square$

Let $\lambda>1$. For any $\delta>0$ and $n\in\mathbb{N}$ define $\mathcal{A}_{\delta,n}(a):=\{0\leqslant j<n\colon\ \|\lambda^j a\|_{\mathbb{Z}}\geqslant\delta\}$. Introduce the sets

$$E_\lambda(n,\varepsilon,\delta)=\bigl\{a\in\mathbb{R}\colon\ \bigl|\mathcal{A}_{\delta,n}(a)\bigr|<\varepsilon n\bigr\}\quad\text{and}$$
$$F_\lambda(n,\varepsilon,\delta)=\bigl\{x/y\colon\ x,y\in E_\lambda(n,\varepsilon,\delta),\ 1\leqslant y<\lambda\bigr\}.$$

Lemma 3.2. *For any $\varepsilon,\delta>0$ and $\ell\in\mathbb{N}$, if $\widehat{f_t}$ does not have uniform decay at infinity then $t\in\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}F_{\lambda^\ell}(n,\varepsilon,\delta)$.*

Proof. Assume that $\widehat{f_t}$ has no uniform decay at infinity. Set $\varepsilon'=\varepsilon/(2\ell)$. Then by Lemma 3.1, for any $N\in\mathbb{N}$ there exist $a\in[1,\lambda]$ and $n>N$ such that $|\mathcal{A}_{\delta,n}(a)|<\varepsilon' n$ and $|\mathcal{A}_{\delta,n}(at)|<\varepsilon' n$. Define $m=[n/\ell]$, where $[x]$ denotes the integral part of x. Observe that

$$\bigl\{0\leqslant j<m\colon\ \bigl\|a\lambda^{\ell j}\bigr\|_{\mathbb{Z}}\geqslant\delta\bigr\}\subset\bigl\{0\leqslant j<n\colon\ \bigl\|a\lambda^{j}\bigr\|_{\mathbb{Z}}\geqslant\delta\bigr\}=\mathcal{A}_{\delta,n}(a).$$

It follows that the cardinality of $\{0\leqslant j<m\colon\ \|a\lambda^{\ell j}\|_{\mathbb{Z}}\geqslant\delta\}$ is not exceeding $\varepsilon' n\leqslant\varepsilon m$. Thus $a\in E_{\lambda^\ell}(m,\epsilon,\delta)$. An identical argument shows $at\in E_{\lambda^\ell}(m,\epsilon,\delta)$. Since $a\in[1,\lambda]$, we have $t\in F_{\lambda^\ell}(m,\epsilon,\delta)$. Noting that m can take infinitely many integers, we obtain $t\in\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}F_{\lambda^\ell}(m,\epsilon,\delta)$. $\square$

Proposition 3.3. *Let $\lambda>16$. There exist $\varepsilon_0,\delta_0>0$ (depending on λ) such that the set $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}F_\lambda(n,\varepsilon_0,\delta_0)\cap[1,\infty)$ has the Hausdorff dimension not exceeding* $\log 16/\log\lambda$.

This proposition is the key ingredient in, and forms the bulk of, the proof of Theorem 1.4. We shall prove Proposition 3.3 later.

Proof of Theorem 1.4. First we consider the case $\lambda>1$. Pick $\ell\in\mathbb{N}$ so that $\lambda^\ell>16$. By Proposition 3.3, for any integer $j\geqslant\ell$, there exist $\varepsilon_j,\ \delta_j>0$ such that the set $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}F_{\lambda^j}(n,\varepsilon_j,\delta_j)\cap[1,\infty)$ has the Hausdorff dimension not exceeding $\log 16/(j\log\lambda)$. Denote

$$E_1=\bigcap_{j\geqslant\ell}\Biggl(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}F_{\lambda^j}(n,\varepsilon_j,\delta_j)\cap[1,\infty)\Biggr).$$

Then $\dim_H E_1=0$ and it is independent of $\{c_j\}$. For any $t>1$ and $t\notin E_1$, $\widehat{f_t}$ has uniform decay at ∞ by Lemma 3.2.

We still need to prove that $\widehat{f_t}$ has uniform decay at ∞ for all $t<1$ except for a set of zero Hausdorff dimension. For $0<t<1$ consider $g(x):=g_t(x)=f(tx)$. Then g satisfies the refinement equation

$$g(x)=\sum_{j=0}^{n}c_j g\bigl(\lambda x-d_j t^{-1}\bigr),$$

which contains the translations $\{0,1,t^{-1}\}$. Let $t_1=t^{-1}$. Then $\widehat{g_t}$ has uniform decay at ∞ for all $t_1>1$ except for a set of zero Hausdorff dimension. Hence there exists an $E_2\subset(0,1)$ with

$\dim_H(E_2) = 0$ such that $\widehat{f_t}$ has uniform decay for all $t \in (0,1) \setminus E_2$. Finally, for $t < 0$ we let $h(x) := h_t(x) = f(-tx + \frac{t}{\lambda-1})$. Then $h = h_t$ satisfies

$$h(x) = \sum_{j=0}^{n} c_j h\big(\lambda x + d_j t^{-1} - 1\big),$$

which has $\{0, 1, 1 - t^{-1}\}$ among the translations. Set $t_2 = 1 - t^{-1}$, and the same argument now yields the existence of $E_3 \subset (-\infty, 0)$ with zero Hausdorff dimension such that $\widehat{f_t}$ has uniform decay for all $t \in (-\infty, 0) \setminus E_3$. The theorem is now proved by letting $E = E_1 \cup E_2 \cup E_3$. E is independent of $\{c_j\}$.

Next, for dilation λ with $\lambda < -1$, we may iterate the refinement equation (1.3) 2 times to obtain a new refinement equation with the same solution, which now has $\lambda^2 > 1$ as its dilation factor. Note that $\{0, 1, t\}$ remain part of the translation set for the new refinement equation. Hence $\widehat{f_t}$ has uniform decay for almost all t except for a set of zero Hausdorff dimension. □

The remaining part of this section is devoted to the proof of Proposition 3.3. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x.

Lemma 3.4. *Let $\lambda > 2$ and $x, y \in \mathbb{R}$. Suppose that $\lambda^{-n-1} \leqslant x - y < \lambda^{-1}$ for some $n \in \mathbb{N}$. Then there exists an integer k with $0 \leqslant k \leqslant n-1$ such that $|\{\lambda^k x\} - \{\lambda^k y\}| \geqslant \lambda^{-2}$.*

Proof. Write $x - y = a\lambda^{-k}$ for $a \in [\lambda^{-2}, \lambda^{-1})$ and $k \in \mathbb{Z}$. From the condition $\lambda^{-(n+1)} \leqslant x - y < \lambda^{-1}$, we obtain $0 \leqslant k \leqslant n-1$. Since $\lambda^k x - \lambda^k y = a$ with $0 < a < \lambda^{-1} \leqslant \frac{1}{2}$, we have $\{\lambda^k x\} - \{\lambda^k y\} = a + l$ for some $l \in \mathbb{Z}$. Thus $|\{\lambda^k x\} - \{\lambda^k y\}| \geqslant a \geqslant \lambda^{-2}$. □

Now for any integer $M > \lambda^2$ and $\mathbf{k} = (k_0, k_1, \ldots, k_{n-1}) \in Z_M^n$, where $Z_M := \{0, 1, \ldots, M-1\}$, denote

$$\Gamma_{T,M}(\mathbf{k}) = \left\{ a \in [1,T]\colon \left\{a\lambda^j\right\} \in \left[\frac{k_j}{M}, \frac{k_j+1}{M}\right) \text{ for } 0 \leqslant j \leqslant n-1 \right\}.$$

It is clear that the collection $\{\Gamma_{T,M}(\mathbf{k})\colon \mathbf{k} \in Z_M^n\}$ is a Borel partition of the interval $[1,T]$.

Lemma 3.5. *Let $\lambda > 2$. For each $n \in \mathbb{N}$ and $\mathbf{k} \in Z_M^n$, the set $\Gamma_{T,M}(\mathbf{k})$ can be covered by at most $4\lambda T + 2$ intervals of length λ^{-n-1}.*

Proof. By the definition of $\Gamma_{T,M}(\mathbf{k})$, for any $x, y \in \Gamma_{T,M}(\mathbf{k})$ we have

$$\left|\left\{\lambda^j x\right\} - \left\{\lambda^j y\right\}\right| \leqslant \frac{1}{M} < \lambda^{-2}, \quad \text{for all } 0 \leqslant j \leqslant n-1.$$

Thus by Lemma 3.4 either $|x - y| \geqslant \lambda^{-1}$ or $|x - y| < \lambda^{-(n+1)}$.

Now define a set $A := \{x \in [1,T]\colon \operatorname{dist}(x, \Gamma_{T,M}(\mathbf{k})) \leqslant \lambda^{-n-1}/2\}$. Then A can be written as $\bigcup_i I_i$, where I_i's are disjoint intervals. Since $|x - y| \geqslant \lambda^{-1}$ or $|x - y| < \lambda^{-(n+1)}$ for any $x, y \in \Gamma_{T,M}(\mathbf{k})$, each interval I_i has length not exceeding $2\lambda^{-(n+1)}$, and the gap between any two intervals I_i, $I_{i'}$ has length greater than $\frac{1}{2}\lambda^{-1}$. Thus the number of different intervals I_i's is less than $2\lambda T+1$. Since each I_i can be covered by at most two intervals of length $\lambda^{-(n+1)}$, we obtain the desired result. $\square$

Lemma 3.6. *Let $\lambda > 16$. There exist $\varepsilon_0, \delta_0 > 0$ such that for any sufficiently large $n \in \mathbb{N}$ and any $T > 1$, the set $E_\lambda(n, \varepsilon_0, \delta_0) \cap [1, T]$ can be covered by at most $4^n(4\lambda T + 2)$ subintervals of $[1, T]$ of length $\lambda^{-(n+1)}$.*

Proof. Pick an integer $M > \lambda^2$. For any $n \in \mathbb{N}$ the collection of sets $\Gamma_{T,M}(\mathbf{k})$ where $\mathbf{k}$ runs through Z_M^n is a Borel partition of the interval $[1, T]$. Take $\delta_0 = \frac{1}{M}$. Notice that $\|x\|_{\mathbb{Z}} < \delta_0$ if and only if $\{x\} \in [0, \frac{1}{M})$ or $\{x\} \in [\frac{M-1}{M}, 1)$. Assume that $0 < \varepsilon < 1/4$. It follows from the definition that for $E_\lambda(n, \varepsilon, 1/M) \cap \Gamma_{T,M}(\mathbf{k}) \neq \emptyset$ we must have

$$\bigl|\bigl\{0 \leqslant j \leqslant n-1\colon\ k_j \notin \{0, M-1\}\bigr\}\bigr| \leqslant \varepsilon n. \tag{3.1}$$

Denote by $[x]$ the integral part of x. Let $\mathcal{B}_{n,\varepsilon}$ denote the collection of all $\mathbf{k} \in Z_M^n$ satisfying (3.1). Then an easy combinatorial argument yields

$$|\mathcal{B}_{n,\varepsilon}| = \sum_{k=0}^{[\varepsilon n]} \binom{n}{k} (M-2)^k 2^{n-k} \leqslant [\varepsilon n] \binom{n}{[\varepsilon n]} M^{[\varepsilon n]} 2^{n+1}.$$

By the Stirling formula,

$$\binom{n}{[\varepsilon n]} = e^{n(-\varepsilon \log \varepsilon - (1-\varepsilon)\log(1-\varepsilon) + o(1))}.$$

It follows that

$$|\mathcal{B}_{n,\varepsilon}| \leqslant [\varepsilon n] M^{\varepsilon n} 2^{n+1} e^{n(-\varepsilon \log \varepsilon - (1-\varepsilon)\log(1-\varepsilon) + o(1))}.$$

Let $\varepsilon = \varepsilon_0$ be small enough so that

$$M^{\varepsilon_0} e^{-\varepsilon_0 \log \varepsilon_0 - (1-\varepsilon_0)\log(1-\varepsilon_0)} < 2.$$

Then $|\mathcal{B}_{n,\varepsilon_0}| \leqslant 4^n$ for sufficiently large n. Since

$$E_\lambda(n, \varepsilon_0, 1/M) \cap [1, T] \subset \bigcup_{\mathbf{k} \in \mathcal{B}_{n,\varepsilon_0}} \Gamma_{T,M}(\mathbf{k}),$$

by Lemma 3.5 $E_\lambda(n, \varepsilon_0, 1/M) \cap [1, T]$ can be covered by at most $4^n(4\lambda T + 2)$ subintervals of $[1, T]$ of length λ^{-n-1}. This proves the lemma. $\square$

Lemma 3.7. *Let $\lambda > 16$. Let $\varepsilon_0, \delta_0 > 0$ be as in Lemma 3.6. Then for any $T > \lambda$ the set $F_\lambda(n, \varepsilon_0, \delta_0) \cap [1, \lambda^{-1}T]$ can be covered by at most $4^{2n}(4\lambda T + 2)^2$ subintervals of $[1, \lambda^{-1}T]$ of length $2T\lambda^{-n-1}$.*

Proof. Observe that

$$F_\lambda(n,\varepsilon_0,\delta_0)\cap\left[1,\lambda^{-1}T\right]\subset\left(E_\lambda(n,\varepsilon_0,\delta_0)\cap[1,T]\right)/\left(E_\lambda(n,\varepsilon_0,\delta_0)\cap[1,\lambda]\right).$$

By Lemma 3.6, for any large enough n the set $E_\lambda(n,\varepsilon_0,\delta_0)\cap[1,T]$ can be covered by at most $4^n(4\lambda T+2)$ subintervals of $[1,T]$ of length $\lambda^{-(n+1)}$. Denote these intervals by I_i, $1\leqslant i\leqslant p$, where p is an integer not exceeding $4^n(4\lambda T+2)$. Set $B_{i,j}=\{x/y\colon\ x\in I_i,\ \ y\in I_j\}$ for $1\leqslant i,j\leqslant p$. Then $B_{i,j}$ is an interval. It is not hard to check that each $B_{i,j}$ has length less than $2T\lambda^{-n-1}$. Now $F_\lambda(n,\varepsilon_0,\delta_0)\cap[1,\lambda^{-1}T]\subset\bigcup_{1\leqslant i,j\leqslant p}B_{i,j}$. This completes the proof of the lemma. □

Proof of Proposition 3.3. By Lemma 3.7, there exist $\varepsilon_0,\delta_0>0$ such that for any fixed $T>\lambda$ and sufficiently large $n\in\mathbb{N}$, the set $F_\lambda(n,\varepsilon_0,\delta_0)\cap[1,\lambda^{-1}T]$ can be covered by at most $4^{2n}(4\lambda T+2)^2$ subintervals of $[1,\lambda^{-1}T]$ of length $2T\lambda^{-n-1}$. It implies immediately that the Hausdorff dimension of $\bigcap_{k=1}^\infty\bigcup_{n=k}^\infty F_\lambda(n,\varepsilon_0,\delta_0)\cap[1,\lambda^{-1}T]$ does not exceed $\log 16/\log\lambda$. The proposition follows since T can be taken arbitrarily large. □

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6 by proving several results concerning the dilation factor λ, where λ is an algebraic number that is the root of an irreducible polynomial $p(x)=\sum_{j=0}^n a_ix^i$ having the property that all other roots of $p(x)$ are outside the unit circle. For convenience we denote $K=|a_0|$.

We divide the proof into two parts. In the first part we prove that $f\in L^\infty$. In the second part we prove $\widehat{f}$ has uniform decay at ∞.

To prove the first part we consider the self-similar measure μ associated with the refinement equation, which is the unique Borel probability measure satisfying the following self-similar relation:

$$\mu=\frac{1}{K}\sum_{j=0}^{K-1}\mu\circ S_j^{-1},\tag{4.1}$$

where $S_j(x)=\lambda^{-1}(x+j)$. It is well known that if μ is absolutely continuous then its density is precisely f. Thus to prove $f\in L^\infty$ we only need to prove that μ has a uniformly bounded density.

Lemma 4.1. *Let $\mathcal{A}:=Z_K-Z_K=\{j\in\mathbb{Z}\colon\ |j|<K\}$. Then $\sum_{j=0}^m\varepsilon_j\lambda^j=0$ for $\varepsilon_0,\dots,\varepsilon_m\in\mathcal{A}$ if and only if $\varepsilon_j=0$ for all j.*

Proof. Let $g(x)=\sum_{j=0}^m\varepsilon_jx^j$. Since $g(\lambda)=0$ and $p(x)=\sum_{j=0}^n a_ix^i$ is the minimal polynomial for λ, it follows that $p(x)\mid g(x)$. Thus $a_0\mid\varepsilon_0$. This yields $\varepsilon_0=0$. Factoring out x in $g(x)$ and repeating the argument yield $\varepsilon_j=0$ for all j. □

Lemma 4.2. *Let $P(x)$ be a polynomial of degree m with integer coefficients. For any $d \in \mathbb{N}$ and variables $x_1, x_2, \ldots, x_d$, set*

$$y_k = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant d} x_{j_1} \cdots x_{j_k}, \quad k = 1, 2, \ldots, d.$$

Then there is an integral polynomial $U(y_1, y_2, \ldots, y_d)$ of degree not exceeding m such that

$$\prod_{j=1}^{d} P(x_j) = U(y_1, y_2, \ldots, y_d).$$

Proof. Note that $\prod_{j=1}^{d} P(x_j)$ is a symmetric polynomial with integer coefficients. It is well known (see e.g. Jacobson [10, Theorem 2.20]) that it can be expressed as $U(y_1, y_2, \ldots, y_d)$ for some integer polynomial U. It remains to prove that $\deg(U) \leqslant m$. To do so we show that $\prod_{j=1}^{d} P(x_j) = U_1(y_1, y_2, \ldots, y_d)$ for some complex polynomial U_1 with $\deg(U_1) \leqslant m$. Then the uniqueness of the polynomial implies that $U = U_1$ (see again [10, Theorem 2.20]).

Now let $P(x) = a(x - \alpha_1) \ldots (x - \alpha_m)$. Then

$$\prod_{j=1}^{d} P(x_j) = a^d \prod_{j=1}^{d} \prod_{k=1}^{m} (x_j - \alpha_k) = a^d \prod_{k=1}^{m} \prod_{j=1}^{d} (x_j - \alpha_k).$$

It is clear that $\prod_{j=1}^{d}(x_j - \alpha_k)$ is a polynomial of $y_1, y_2, \ldots, y_d$ of degree 1. Hence $\prod_{j=1}^{d} P(x_j) = U_1(y_1, y_2, \ldots, y_d)$ for some complex polynomial of degree $\leqslant m$. This proves the lemma. □

Lemma 4.3. *There exists a constant $C > 0$ such that for any $m \in \mathbb{N}$,*

$$\inf\left\{\left|\sum_{j=0}^{m} \varepsilon_j \lambda^j\right| \neq 0\colon\ \varepsilon_j \in \mathcal{A}\right\} \geqslant C|\lambda|^m K^{-m}. \tag{4.2}$$

Proof. Let $\lambda_1 = \lambda$ and $\lambda_2, \ldots, \lambda_n$ be the algebraic conjugates of λ. For any $\varepsilon_0, \ldots, \varepsilon_m \in \mathcal{A}$ set $P(x) = \sum_{j=0}^{m} \varepsilon_j x^j$. By Lemma 4.2, $\prod_{j=1}^{d} P(x_j)$ can be written as $U(y_1, y_2, \ldots, y_n)$ for some integral polynomial U of degree not exceeding m, where y_k are given in Lemma 4.2. Now set $x_j = \lambda_j$. Then elementary algebra tells us that $y_k \in \frac{1}{a_n}\mathbb{Z}$ for all k, so we have $P(\lambda_1)P(\lambda_2) \cdots P(\lambda_n) \in \frac{1}{(a_n)^m}\mathbb{Z}$.

Now assume that $P(\lambda_1) \neq 0$. Then $P(\lambda_k) \neq 0$ for all $2 \leqslant k \leqslant n$. Thus

$$\left|P(\lambda_1)P(\lambda_2) \ldots P(\lambda_n)\right| \geqslant \frac{1}{|a_n|^m}. \tag{4.3}$$

Notice that for any j we have

$$\left|P(\lambda_j)\right| \leqslant (K-1)\left(1 + |\lambda_j| + \cdots + |\lambda_j|^m\right) \leqslant D|\lambda_j|^m$$

for a constant $D > 0$. It follows from (4.3) and the fact $|\prod_{j=1}^{n} \lambda_j| = |a_0/a_n|$ that

$$\begin{aligned}|P(\lambda_1)| &\geqslant \frac{1}{|a_n|^m D^{n-1} \prod_{j=2}^n |\lambda_j|^m} \\ &= \frac{\lambda_1^m}{|a_n|^m D^{n-1} \prod_{j=1}^n |\lambda_j|^m} \\ &= \frac{\lambda_1^m}{D^{n-1}|a_0|^m} \\ &= C|\lambda|^m K^{-m}. \qquad \square\end{aligned}$$

Proposition 4.4. *The self-similar measure μ is absolutely continuous with a bounded density function.*

Proof. Let Δ be the support of μ. It suffices to prove that there exists a constant $M > 0$ such that $\mu(I) \leqslant M|I|$ for any subinterval I of Δ, where $|I|$ denotes the length of I. To do so we write $\mathcal{B} = \{0, 1, \ldots, K-1\}$ and let $\mathcal{B}_m$ denote the set of all words of length m over $\mathcal{B}$. For simplicity, we write $S_{\mathbf{j}} = S_{j_0} \circ S_{j_1} \circ \cdots \circ S_{j_{m-1}}$ for $\mathbf{j} = j_0 j_1 \ldots j_{m-1} \in \mathcal{B}_m$.

Now iterating (4.1) m times yields

$$\mu(I) = \frac{1}{K^m} \sum_{\mathbf{j}\in\mathcal{B}_m} \mu \circ S_J^{-1}(I)$$

for any interval $I = [a, b] \subset \Delta$. Since μ is supported on Δ, it follows that

$$\mu(I) \leqslant K^{-m} \bigl|\{\mathbf{j} \in \mathcal{B}_m\colon\ S_{\mathbf{j}}(\Delta) \cap I \neq \emptyset\}\bigr|. \tag{4.4}$$

Note that $S_{\mathbf{j}}(\Delta)$ has diameter $|\lambda|^{-m}|\Delta|$ where $|\Delta|$ denotes the diameter of Δ. Thus $S_{\mathbf{j}}(\Delta) \cap I \neq \emptyset$ implies $S_{\mathbf{j}}(0) \in [a - |\lambda|^{-m}|\Delta|, b + |\lambda|^{-m}|\Delta|]$, where $S_{\mathbf{j}}(0) = \sum_{k=0}^{m-1} j_k \lambda^{-k}$. Hence by Lemma 4.3, $|S_{\mathbf{j}}(0) - S_{\mathbf{j}'}(0)| \geqslant CK^{-m}$ for different indices $\mathbf{j}, \mathbf{j}' \in \mathcal{B}_m$. Thus for any large integer m we must have

$$\bigl|\bigl\{\mathbf{j} \in \mathcal{B}_m\colon\ S_{\mathbf{j}}(\Delta) \cap I \neq \emptyset\bigr\}\bigr| \leqslant C^{-1}K^m\bigl(|I| + 2|\lambda|^{-m}|\Delta|\bigr) + 1 \leqslant 2C^{-1}K^m|I|.$$

Combining it with (4.4) yields $\mu(I) \leqslant 2C^{-1}|I|$. This completes the proof of the proposition. $\square$

We now turn to the proof of the second part of Theorem 1.6, namely $\widehat{f}$ has uniform decay at infinity.

Lemma 4.5. *Let $H(\xi) = \frac{1}{K}\sum_{j=0}^{K-1} e(-j\xi)$. Suppose $\xi \in \mathbb{R}$ satisfies $\|\xi\|_{\mathbb{Z}} > 1/(2K)$ and $\|K\xi\|_{\mathbb{Z}} \leqslant 1/4$. Then $|H(\xi)| \leqslant 4\|K\xi\|_{\mathbb{Z}}$.*

Proof. Clearly we can write $K\xi = q + \|K\xi\|_{\mathbb{Z}}$ for some $q \in \mathbb{Z}$, and since $\|\xi\|_{\mathbb{Z}} > 1/(2K)$ we have $\xi = p + \frac{j}{K} + \frac{1}{K}\|K\xi\|_{\mathbb{Z}}$ for some $p \in \mathbb{Z}$ and $j \in \{1, \ldots, K-1\}$. Notice that $|H(\xi)| = \frac{|\sin(K\pi\xi)|}{K|\sin(\pi\xi)|}$. Thus

$$\bigl|H(\xi)\bigr| = \frac{|\sin(\pi\|K\xi\|_{\mathbb{Z}})|}{K|\sin(\pi\|\xi\|_{\mathbb{Z}})|}.$$

The lemma is proved from the inequalities $|\sin(\pi\|K\xi\|_{\mathbb{Z}})| \leqslant \pi\|K\xi\|_{\mathbb{Z}}$ and $K|\sin(\pi\|\xi\|_{\mathbb{Z}})| \geqslant K\pi\frac{1}{4K} = \frac{\pi}{4}$. □

Lemma 4.6. *There exist $\ell \in \mathbb{N}$ and integers $b_0, b_1, \ldots, b_\ell$ with $|b_0| > \sum_{j=1}^{\ell}|b_j|$, such that λ is a root of the polynomial $\sum_{j=0}^{\ell} b_j x^j$.*

Proof. The lemma is obviously true when the degree of $p(x)$, the minimal polynomial of λ, is equal to one. So we assume that the degree of λ is larger than 1.

Denote $\lambda_1 = \lambda$ and let $\lambda_2, \ldots, \lambda_n$ be the algebraic conjugates of λ. Then for any $m \in \mathbb{N}$, λ_j^m $(j = 1, \ldots, n)$ are roots of an integral polynomial $P_m(x) = \sum_{j=0}^{n} a_{j,m} x^j$. Since $|\lambda_j| > 1$ for all j, for sufficiently large k we have $|\lambda_j|^k > 2^n$ for all j. For such a k we have

$$\left|\prod_{j=1}^{n} \lambda_j^k\right| > \sum_{u=1}^{n-1}\left|\sum_{1\leqslant j_1<\cdots<j_u\leqslant n} \lambda_{j_1}^k \ldots \lambda_{j_u}^k\right|.$$

The above inequality implies $|a_{0,k}| > \sum_{j=1}^{n}|a_{j,k}|$. Since λ is a root of the polynomial $P_k(x^k) = \sum_{j=0}^{n} a_{j,k} x^{kj}$, we obtain the desired result. □

The following proposition is the key to complete the proof of our theorem.

Proposition 4.7. *There exist $0 < \varepsilon < 1/(4K)$, $0 < \rho < 1$ and $N \in \mathbb{N}$ such that for any $\xi \in \mathbb{R}$ with $|\xi| > |\lambda|^N$, if $\|\lambda^{-j}\xi\|_{\mathbb{Z}} < \varepsilon$ for $j = 1, \ldots, N$, then there exists an integer $m > N$ such that*

$$\left\|\lambda^{-m}\xi\right\|_{\mathbb{Z}} > \frac{1}{2K} \quad \text{and} \quad \left\|K\lambda^{-m}\xi\right\|_{\mathbb{Z}} \leqslant \frac{\rho^m}{4}. \tag{4.5}$$

Proof. Recall that λ is a root of the irreducible integral polynomial $\sum_{j=0}^{n} a_j x^j$. By Lemma 4.6, it is also a root of an integral polynomial $\sum_{j=0}^{\ell} b_j x^j$ with $|b_0| \geqslant \sum_{j=1}^{\ell}|b_j|$. Without loss of generality we assume that $a_0 > 0$ and $b_0 > 0$. Clearly,

$$-1 = \frac{1}{a_0}\sum_{j=1}^{n} a_j\lambda^j \quad \text{and} \quad -1 = \frac{1}{b_0}\sum_{j=1}^{\ell} b_j\lambda^j.$$

Now for any $x \in \mathbb{R}$ let $[\![x]\!]$ denote the integer that is closest to x (especially let $[\![n+1/2]\!] = n$ for $n \in \mathbb{Z}$), and let $]\!]x[\![= x - [\![x]\!]$. Clearly we have $|]\!]x[\![| = \|x\|_{\mathbb{Z}}$. Thus for any $x \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$-\lambda^{-k}x = \frac{1}{a_0}\sum_{j=1}^{n} a_j\lambda^{-k+j}x = \frac{1}{a_0}\sum_{j=1}^{n} a_j[\![\lambda^{-k+j}x]\!] + \frac{1}{a_0}\sum_{j=1}^{n} a_j]\!]\lambda^{-k+j}x[\![\tag{4.6}$$

and

$$-\lambda^{-k}x = \frac{1}{b_0}\sum_{j=1}^{\ell} b_j\lambda^{-k+j}x = \frac{1}{b_0}\sum_{j=1}^{\ell} b_j[\![\lambda^{-k+j}x]\!] + \frac{1}{b_0}\sum_{j=1}^{\ell} b_j]\!]\lambda^{-k+j}x[\![. \tag{4.7}$$

For convenience, we write

$$A_k(x) = \frac{1}{a_0}\sum_{j=1}^{n} a_j \llbracket \lambda^{-k+j} x \rrbracket, \qquad B_k(x) = \frac{1}{a_0}\sum_{j=1}^{n} a_j \rrbracket \lambda^{-k+j} x \llbracket$$

and

$$C_k(x) = \frac{1}{b_0}\sum_{j=1}^{\ell} b_j \llbracket \lambda^{-k+j} x \rrbracket, \qquad D_k(x) = \frac{1}{b_0}\sum_{j=1}^{\ell} b_j \rrbracket \lambda^{-k+j} x \llbracket.$$

It is clear that $A_k(x) \in \frac{1}{a_0}\mathbb{Z}$, $C_k(x) \in \frac{1}{b_0}\mathbb{Z}$ and (4.6), (4.7) can be rewritten as

$$-\lambda^{-k}x = A_k(x) + B_k(x) = C_k(x) + D_k(x). \tag{4.8}$$

Denote $\eta_1 = \frac{1}{a_0}\sum_{j=1}^{n} |a_i|$ and $\eta_2 = \frac{1}{b_0}\sum_{j=1}^{\ell} |b_j|$. Then $0 < \eta_2 < 1$. We shall choose an $\varepsilon > 0$ that is sufficiently small.

Now fix $\xi \in \mathbb{R}$ so that $|\xi| \geqslant |\lambda|^\ell$ and $\|\lambda^{-j}\xi\|_{\mathbb{Z}} < \varepsilon$ for all $1 \leqslant j \leqslant \ell$. In the following we prove that there exist a $\rho \in (0,1)$ (independent of ξ) and an integer $m > \ell$ such that $\|\lambda^{-m}\xi\|_{\mathbb{Z}} > \frac{1}{2a_0}$ and $\|a_0\lambda^{-m}\xi\|_{\mathbb{Z}} < \rho^m/4$.

We first claim that there exists an integer $m > \ell$ such that $C_m(x) \notin \mathbb{Z}$. Assume on the contrary that the claim is not true. Then $C_k(\xi) \in \mathbb{Z}$ for any integer $k > \ell$. By (4.8), for all $k \geqslant \ell + 1$ we have

$$\begin{aligned} \|\lambda^{-k}\xi\|_{\mathbb{Z}} \leqslant |D_k(\xi)| &\leqslant \frac{1}{b_0}\sum_{j=1}^{\ell} |b_j| \|\lambda^{-k+j}\xi\|_{\mathbb{Z}} \\ &\leqslant \eta_2 \max\{\|\lambda^{-k+j}\xi\|_{\mathbb{Z}}\colon 1 \leqslant j \leqslant \ell\}. \end{aligned} \tag{4.9}$$

Since $\|\lambda^{-j}\xi\|_{\mathbb{Z}} < \varepsilon$ for $1 \leqslant j \leqslant \ell$, by (4.9) and an inductive argument we have $\|\lambda^{-k}\xi\|_{\mathbb{Z}} < \varepsilon$ for all $k \geqslant \ell + 1$. However since $|\xi| > |\lambda|^\ell$, there exists some integer $k_0 > \ell$ such that $|\lambda^{-k_0}\xi| \in [|\lambda|^{-2}, |\lambda|^{-1})$. Therefore with ε small we have $\|\lambda^{-k_0}\xi\|_{\mathbb{Z}} \geqslant \min\{\|\lambda^{-2}\|_{\mathbb{Z}}, \|\lambda^{-1}\|_{\mathbb{Z}}\} > \varepsilon$, which leads to a contradiction. This finishes the claim.

Now assume without loss of generality that m is the smallest integer so that $m > \ell$ and $C_m(\xi) \notin \mathbb{Z}$. We consider the following two cases separately.

Case 1. $m = \ell + 1$.

In this case we have

$$|D_{\ell+1}(\xi)| \leqslant \eta_2 \max\{\|\lambda^{-(\ell+1-j)}\|_{\mathbb{Z}}\colon 1 \leqslant j \leqslant \ell\} \leqslant \eta_2\varepsilon,$$

and with ε small enough,

$$\|\lambda^{-\ell-1}\xi\|_{\mathbb{Z}} \geqslant \|C_{\ell+1}(\xi)\|_{\mathbb{Z}} - \|D_{\ell+1}(\xi)\|_{\mathbb{Z}} \geqslant \frac{1}{b_0} - \eta_2\varepsilon \geqslant \frac{1}{2b_0}.$$

Case 2. $m \geqslant \ell + 2$.

Write $m = 2 + p\ell + q$, where $p \in \mathbb{N}$ and $0 \leqslant q \leqslant \ell - 1$. Since $C_k(\xi) \in \mathbb{Z}$ for $\ell + 1 \leqslant k \leqslant m - 1$, as with (4.9) we have

$$\|\lambda^{-k}\xi\|_{\mathbb{Z}} \leqslant \eta_2 \max\{\|\lambda^{-k+j}\xi\|_{\mathbb{Z}}\colon 1 \leqslant j \leqslant \ell\} \quad \text{for all } \ell + 1 \leqslant k \leqslant m - 1. \tag{4.10}$$

Using (4.10) and induction we have

$$\|\lambda^{-t\ell-j}\xi\|_{\mathbb{Z}} \leqslant \eta_2^t \varepsilon$$

for any integers t, j such that $1 \leqslant t \leqslant p$, $1 \leqslant j \leqslant \ell$ and $t\ell + j \leqslant m - 1$. Particularly

$$\|\lambda^{-m+j}\xi\|_{\mathbb{Z}} \leqslant \eta_2^{p-1}\varepsilon, \quad j = 1, 2, \ldots, \ell.$$

Thus

$$|D_m(\xi)| \leqslant \eta_2 \max\{\|\lambda^{-(m-j)}\xi\|_{\mathbb{Z}}\colon 1 \leqslant j \leqslant \ell\} \leqslant \eta_2^p \varepsilon \leqslant \eta_2^{\frac{m}{2\ell}} \varepsilon$$

and with ε small enough,

$$\|\lambda^{-m}\xi\|_{\mathbb{Z}} \geqslant \|C_m(\xi)\|_{\mathbb{Z}} - \|D_m(\xi)\|_{\mathbb{Z}} \geqslant \frac{1}{b_0} - \varepsilon \geqslant \frac{1}{2b_0}.$$

Take $\rho = \eta_2^{\frac{1}{2\ell}}$. We have proved that in each case there always exists an integer $m \geqslant \ell + 1$ such that $\|\lambda^{-m}\xi\|_{\mathbb{Z}} \geqslant \frac{1}{2b_0}$, $\|b_0\lambda^{-m}\xi\|_{\mathbb{Z}} \leqslant b_0\rho^m\varepsilon$ and $\|\lambda^{-m+j}\xi\|_{\mathbb{Z}} < \varepsilon$ for all $1 \leqslant j \leqslant \ell$. Observe that

$$|B_m(\xi)| \leqslant \eta_1 \max\{\|\lambda^{-(m-j)}\xi\|_{\mathbb{Z}}\colon 1 \leqslant j \leqslant \ell\} \leqslant \eta_1\varepsilon.$$

From the fact $A_m(\xi) + B_m(\xi) = C_m(\xi) + D_m(\xi)$ we have

$$|A_m(\xi) - C_m(\xi)| = |B_m(\xi) - D_m(\xi)| \leqslant (\eta_1 + 1)\varepsilon < \frac{1}{a_0 b_0}.$$

It implies that $A_m(\xi) - C_m(\xi) = 0$ since $A_m(\xi) \in \frac{1}{a_0}\mathbb{Z}$ and $C_m(\xi) \in \frac{1}{b_0}\mathbb{Z}$. Hence $B_m(\xi) = D_m(\xi)$. By making ε small it follows that

$$\|a_0\lambda^{-m}\xi\|_{\mathbb{Z}} \leqslant |a_0 B_m(\xi)| = |a_0 D_m(\xi)| \leqslant a_0\rho^m\varepsilon \leqslant \frac{\rho^m}{4}.$$

Since $C_m(\xi) \notin \mathbb{Z}$, so does $A_m(\xi)$. Hence $\|A_m(\xi)\|_{\mathbb{Z}} \geqslant \frac{1}{a_0}$,

$$\|\lambda^{-m}\xi\|_{\mathbb{Z}} \geqslant \|A_m(\xi)\|_{\mathbb{Z}} - \|B_m(\xi)\|_{\mathbb{Z}} \geqslant \frac{1}{a_0} - \eta_1\varepsilon > \frac{1}{2a_0}.$$

This finishes the proof of the proposition. $\square$

Proof of Theorem 1.6. Let f be the unique compactly supported distribution of (1.4) with $\widehat{f}(0) = 1$. Let ε, ρ and N be given in Proposition 4.7. To prove that $\widehat{f}$ has a uniform decay at infinity, we only need to show that there exists a $\delta > 0$ such that for any $\xi \in \mathbb{R}$ with $|\xi| > |\lambda|^N$, there exists an $\ell \in \mathbb{N}$ such that

$$\left|\widehat{f}(\xi)\right| \leqslant \lambda^{-\ell\delta}\left|\widehat{f}(\lambda^{-\ell}\xi)\right|.$$

Now $\widehat{f}(\xi) = \widehat{f}(\xi/\lambda)H(\xi/\lambda)$, where $H(\xi)$ is defined as in Lemma 4.5. For any $k \in \mathbb{N}$, iterating the above equality k times yields $\widehat{f}(\xi) = \widehat{f}(\lambda^{-k}\xi)H(\lambda^{-1}\xi)\dots H(\lambda^{-k}\xi)$. Using the inequality $|H(x)| \leqslant 1$ we have

$$\left|\widehat{f}(\xi)\right| \leqslant \left|\widehat{f}\big(\lambda^{-k}\xi\big)\right|\left|H\big(\lambda^{-k}\xi\big)\right|, \quad \forall \xi \in \mathbb{R},\ k \in \mathbb{N}. \tag{4.11}$$

Now set $C := \max\{|H(\xi)|: \|\xi\|_{\mathbb{Z}} \geqslant \varepsilon\}$. It is clear $0 < C < 1$. According to Proposition 4.7, for any $\xi \in \mathbb{R}$ with $|\xi| > |\lambda|^N$, either $\|\lambda^{-j}\xi\|_{\mathbb{Z}} \geqslant \varepsilon$ for some $1 \leqslant j \leqslant N$, or there exists $m > N$ such that $\|\lambda^{-m}\xi\|_{\mathbb{Z}} > \frac{1}{2K}$ and $\|K\lambda^{-m}\xi\|_{\mathbb{Z}} < \rho^m/4$. By Lemma 4.5, either $|H(\lambda^{-j}\xi)| \leqslant C$ for some $1 \leqslant j \leqslant N$, or $|H(\lambda^{-m}\xi)| \leqslant 4\|K\lambda^{-m}\xi\|_{\mathbb{Z}} \leqslant \rho^m$ for some $m > N$. Define

$$\delta = \min\left\{\frac{\log(1/C)}{N\log\lambda},\ \frac{\log(1/\rho)}{\log\lambda}\right\}.$$

Then for any $\xi \in \mathbb{R}$ with $|\xi| > \lambda^N$, there exist $\ell \in \mathbb{N}$ such that $H(\lambda^{-\ell}\xi) \leqslant \lambda^{-\ell\delta}$. The theorem now follows from (4.11). □

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Appendix A. Open questions and results of Erdös and Kahane

In this section we prove that compactly supported refinable distributions with integer translations do not have uniform decay at infinity if the dilations are Pisot or Salem numbers. This result was established in the case of Bernoulli convolutions by Erdös ([6], Pisot numbers) and Kahane ([11], Salem numbers). The general case stated here is proved using Kahane's technique, although some nontrivial technical details had to be overcome.

Proposition A.1. *Let $f(x)$ be the refinable distribution given by*

$$f(x) = \sum_{j=0}^{n} c_j f(\lambda x - d_j), \quad \sum_{j=0}^{n} c_j = |\lambda|, \tag{A.1}$$

where λ is a Pisot number or Salem number with $|\lambda| > 1$ and $\lambda \notin \mathbb{Z}$, $c_j \neq 0$ and $d_j \in \mathbb{Z}$ for all j. Then $\widehat{f}$ does not have uniform decay at infinity.

To prove this proposition, we need the following result, which was first proved by Pisot (the reader may see [15] for a proof).

Theorem A.2. *(Pisot [14].) Let λ be an arbitrary algebraic integer. Then there exists a Pisot number in $\mathbb{Z}[\lambda]$ having the same degree as λ.*

Corollary A.3. *Let λ be a Pisot or Salem number. Then there exists a sequence $\{u_m\}$ in $\mathbb{Z}[\lambda]$ such that $\|\lambda^k u_m\|_{\mathbb{Z}} < \frac{1}{m}$ for all $m, k \in \mathbb{N}$. Furthermore, if λ is a Pisot number then we may take $u_m = m\lambda^{\alpha_m}$ for some $\alpha_m \in \mathbb{N}$. If λ is a Salem number then we may take $u_m = \omega^{\alpha_m}$ for some $\alpha_m \in \mathbb{N}$, where $\omega \in \mathbb{Z}[\lambda]$ is a Pisot number independent of m.*

Proof. Assume that λ is of degree d. Let $\lambda_1, \ldots, \lambda_{d-1}$ be the algebraic conjugates of λ. Observe that if $f(x) \in \mathbb{Z}[x]$ then $f(\lambda) + \sum_{j=1}^{d-1} f(\lambda_j) \in \mathbb{Z}$. In particular $\|f(\lambda)\|_{\mathbb{Z}} \leqslant \sum_{j=1}^{d-1} |f(\lambda_j)|$.

Suppose λ is a Pisot number. Then $|\lambda_j| < 1$. Choose $\alpha_m > 0$ so that $\sum_{j=1}^{d-1} m|\lambda_j|^{\alpha_m} < \frac{1}{m}$. Then

$$\left\|\lambda^k u_m\right\|_{\mathbb{Z}} = \left\|m\lambda^{k+\alpha_m}\right\|_{\mathbb{Z}} \leqslant \sum_{j=1}^{d-1} m|\lambda_j|^{\alpha_m+k} \leqslant \frac{1}{m}.$$

Suppose λ is a Salem number. Let $\omega = f(\lambda)$ be a Pisot number of the same degree as λ, where $f(x) \in \mathbb{Z}[x]$. Then the algebraic conjugates of ω are $\omega_j := f(\lambda_j)$. Let $\alpha_m > 0$ such that $\sum_{j=1}^{d-1} |f(\lambda_j)|^{\alpha_m} < \frac{1}{m}$. It follows from the property that $|\lambda_j| \leqslant 1$ that

$$\left\|\lambda^k u_m\right\|_{\mathbb{Z}} = \left\|\lambda^k \omega^{\alpha_m}\right\|_{\mathbb{Z}} \leqslant \sum_{j=1}^{d-1} |\lambda_j|^k \left|f(\lambda_j)\right|^{\alpha_m} \leqslant \frac{1}{m}. \qquad \square$$

Proof of Proposition A.1. Assume that Proposition A.1 is not true. Then $\widehat{f}$ has a γ-uniform decay for some $\gamma > 0$. Let $P(\xi)$ be the symbol of the refinement equation (A.1). Since $P(n) = 1$ for any integer n, we may choose m_0 sufficiently large so that $|P(\lambda^k u_m)| > |\lambda|^{-\gamma/2}$ for all $m > m_0$ and $k \geqslant 0$, where u_m is as in Corollary A.3. We claim that $\widehat{f}(u_m) = 0$ for all $m > m_0$. Assume it is false, i.e. $\widehat{f}(u_m) \neq 0$ for some $m > m_0$. Then for any $N \in \mathbb{N}$,

$$\left|\widehat{f}\left(\lambda^N u_m\right)\right| = \left|\widehat{f}(u_m)\right| \prod_{j=0}^{N-1} \left|P\left(\lambda^j u_m\right)\right| \geqslant |\lambda|^{-\gamma N/2} \widehat{f}(u_m).$$

However the above inequality contradicts the fact that $\widehat{f}$ has a γ-uniform decay. This proves the claim.

Now $\widehat{f}(u_m) = 0$ implies that there exists a $j_m \in \mathbb{N}$ such that $P(u_m\lambda^{-j_m}) = 0$. We now consider the case that λ is a Pisot number. In this case $u_m = m\lambda^{\alpha_m}$. Set $k_m = j_m - \alpha_m$. Then $P(m\lambda^{-k_m}) = 0$ for all $m > m_0$. Let K be an integer such that $\lambda^{-1} \in \frac{1}{K}\mathbb{Z}[\lambda]$. Then $\lambda^{-k_m} \in K^{-k_m}\mathbb{Z}[\lambda]$. Write

$$\lambda^{-k_m} = K^{-k_m}\left(p_{m,0} + p_{m,1}\lambda + \cdots + p_{m,d-1}\lambda^{d-1}\right),$$

where d is the degree of λ and $p_{m,i} \in \mathbb{Z}$ for all i. This expression is unique since $\{\lambda^i\colon 0 \leqslant i < d\}$ are linearly independent over $\mathbb{Q}$. Hence

$$m\lambda^{-k_m} = K^{-k_m}\left(mp_{m,0} + mp_{m,1}\lambda + \cdots + mp_{m,d-1}\lambda^{d-1}\right).$$

Since $\lambda^{-k_m} \notin \mathbb{Q}$, at least one of $p_{m,i} \neq 0$ for some $1 \leqslant i \leqslant d-1$. Now $P(\xi)$ is a trigonometric polynomial, so $\{m\lambda^{-k_m} \pmod 1\colon\ m > m_0\}$ is a finite set. Thus again by the linear independence of $\{\lambda^i\colon\ 0 \leqslant i < d\}$ over $\mathbb{Q}$ we know that the set

$$\big\{\big(K^{-k_m} m p_{m,1}, \ldots, K^{-k_m} m p_{m,d-1}\big)\colon\ m > m_0\big\}$$

is a finite set. But this is not true, because we may take m sufficiently large and coprime with K so that the nonzero numerators in $(K^{-k_m} m p_{m,1}, \ldots, K^{-k_m} m p_{m,d-1})$ become arbitrarily large. This yields a contradiction.

Next we consider the case that λ is a Salem number. In this case, $u_m = \omega^{\alpha_m}$ where $\omega = f(\lambda) \in \mathbb{Z}[\lambda]$ is a Pisot number. Hence for each $m > m_0$ we have $P(\lambda^{-j_m}\omega^{\alpha_m}) = 0$ for some $j_m > 0$. Again, $\{\lambda^{-j_m}\omega^{\alpha_m} \pmod 1\colon\ m > m_0\}$ is a finite set. Choose m, n such that $\alpha_m \neq \alpha_n$ are sufficiently large and

$$\lambda^{-j_m}\omega^{\alpha_m} - \lambda^{-j_n}\omega^{\alpha_n} = \ell \in \mathbb{Z}.$$

Without loss of generality assume that $j_m \geqslant j_n$. Then $\omega^{\alpha_m} - \lambda^{j_m - j_n}\omega^{\alpha_n} = \ell\lambda^{j_m}$. However, this is a contradiction because for sufficiently large α_m, α_n the left-hand side is a Pisot number but the right-hand side is not. This completes the proof of the proposition. $\square$

There are a number of interesting open questions. We list some of them here.

(1) Is it true that uniform decay in $\widehat{f}$ (assuming integer translations) can always be achieved for dilations that are not Pisot or Salem numbers? A related questions is whether Pisot numbers are the only ones that give singular Bernoulli convolutions. This question is known to be hard.
(2) Can one find another family of dilations for which refinable functions with uniform decay property in $\widehat{f}$ can be constructed explicitly?
(3) Fix the translations and weights (nonnegative) of a refinement equation, is it true that by varying the dilation λ the resulting refinable distribution has uniform decay in $\widehat{f}$ for almost all $\lambda > 1$?
(4) The algebraic properties of the dilation factor λ play key roles in the study of refinable functions. In Dai et al. [2] it was shown that the dilation factor for a refinable spline with integer translations must be the root of an integer. It is an interesting problem to find other such connections.

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